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**LETTER TO THE EDITOR**

# A note on quasi-periodic states†

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**Abstract.** We point out some subtleties in the existence problem of quasi-periodic states for quantum Hamiltonians periodic in time.

The interaction of light with atomic electrons was one of the phenomena which led to the development of quantum mechanics. Since electromagnetic waves are periodic in time, the description requires the study of time-dependent periodic Hamiltonians $H(t+\tau)=H(t)$, $\tau>0$. If the intensity of the wave is low enough, one obtains a satisfactory description by perturbation theory, starting from a time-independent Hamiltonian. Since the advent of lasers with high intensities this approach no longer suffices. Multi-photon spectroscopy has developed rapidly into a separate field of research. An immense literature is available, both on experimental and theoretical developments, and the progress is recorded at special congresses (see for example Eberly and Karczewski 1977, Eberly and Lambropoulos 1978, Eberly *et al* 1979). Despite this activity and the practical importance of the field, so far there have been only a few rigorous investigations of quantum mechanics with Hamiltonians periodic in time. A concept which seems appropriate for a rigorous treatment is that of quasi-periodic states (Zeldovich 1967). This concept underlies many practical applications and has attracted much interest (see Salzman 1974a and references therein). If the underlying state space is finite dimensional, the existence of quasi-periodic (QP) states is generally accepted (Shirley 1965, Young *et al* 1969, Salzman 1974a) but for infinite dimensions precise statements are missing. In fact, some authors simply postulated their existence (Sambe 1973), whereas others (Young *et al* 1969) developed arguments against them.

In this Letter we discuss these existence problems on the basis of elementary functional analytic considerations and present a concise but rigorous treatment of QP states and their main properties.

Throughout this Letter we assume the following conditions to be fulfilled.

(A1) $H(t)$, $t\in R$ is a family of self-adjoint periodic Hamiltonians in a Hilbert space $\mathscr{H}$, $H(t+\tau)=H(t)$, $\tau>0$, for all $t\in R$.

(A2) There exists an evolution operator $U(t, t_0)$, $t, t_0\in R$, with the properties:

(a) $U(t, t_0)$ is unitary for all $t, t_0\in R$,

(b) $U(t, t)=1$, $\qquad t\in R$,

(c) $U(t, t_1)U(t_1, t_2)=U(t, t_2)$, $\qquad t, t_1, t_2\in R$,

(d) $U(t+\tau, t_0+\tau)=U(t, t_0)$, $\qquad t, t_0\in R$,

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(e) $U(t, t_0)$ is strongly continuous in $t$ and $t_0$,
(f) if $f_0 \in \text{dom}[H(t_0)]$ for some $t_0 \in R$, then $U(t, t_0)f_0 \in \text{dom}[H(t)]$ for all $t \in R$ and

$$\frac{d}{dt} U(t, t_0)f_0 = -iH(t)U(t, t_0)f_0, \qquad t \in R.$$

Sufficient conditions on $H(t)$ in order that (A2(a)–(f)) are valid are well known (Simon 1971, Kato 1973, Prugovecki and Tip 1974, Yajima 1977, Howland 1979) and need not be reproduced here.

Without loss of generality we put $t_0 = 0$ and note that (A1) and (A2) immediately imply the Floquet form (Shirley 1965, Salzman 1974a, Barone *et al* 1977) of $U(t) \equiv U(t, 0)$.

*Proposition 1.* Assume (A1) and (A2(a)–(f)). Then $U(t)$ may be written as

$$U(t) = P(t)\, e^{-itG}, \qquad t \in R, \tag{1}$$

where $G$ is self-adjoint, and $P(t)$ is unitary and periodic, $P(t+\tau) = P(t)$, $t \in R$. For a proof see Salzman (1974a) and Barone *et al* (1977).

Note that unitarity of $U(t)$ (which is responsible for the self-adjointness of $G$) implies that the spectrum $\sigma(G)$ of $G$ is real, $\sigma(G) \subseteq R$, which is in contrast to the situation in Hill's equation where characteristic exponents are in general complex (Becker *et al* 1979).

Now we are in a position to discuss QP states associated with $H(t)$. They are usually introduced as solutions of

$$\frac{d}{dt} \Phi(\lambda, t) = -iH(t)\Phi(\lambda, t), \qquad \Phi(\lambda, t) \in \text{dom}[H(t)], \tag{2}$$

with the additional property

$$\Phi(\lambda, t+\tau) = e^{-i\tau\lambda}\, \Phi(\lambda, t), \qquad \lambda \in R. \tag{3}$$

Here the parameter $\lambda$, which is unique up to multiples of $2\pi n/\tau$, $n \in Z$, is called quasi-energy (Ritus 1967).

In order to reduce the complexity of the problem, we eliminate the $t$ dependence completely and consider the related eigenvalue problem

$$G\psi(\lambda) = \lambda\psi(\lambda), \qquad \psi(\lambda) \in \text{dom}(G). \tag{4}$$

If $\phi(\lambda) \in \text{dom}(G) \cap \text{dom}[H(0)]$ fulfils (4) then

$$\Phi(\lambda, t) = U(t)\phi(\lambda)$$

is easily seen to be a solution of (2) and (3). On the other hand, if $\Psi(\mu, t) \in \text{dom}[H(t)]$ fulfils (2) and (3), then $\Psi(\mu, 0)$ fulfils

$$U(\tau)\Psi(\mu, 0) = e^{-i\tau G}\, \Psi(\mu, 0) = e^{-i\tau\mu}\, \Psi(\mu, 0)$$

and hence $\Psi(\mu, 0)$ is a solution of (4) with $\lambda = \mu + 2\pi n/\tau$ for some $n \in Z$.

Given the eigenvalue problem (4) we can thus answer the existence question of QP states in the following way.

*Proposition 2.* Suppose (A1) and (A2(a)–(f)) are fulfilled. Then QP states associated with $H(t)$ exist if and only if $G$ has eigenstates $\psi(\lambda)$ contained in the domain of $H(0)$. The QP states are then given by $U(t)\psi(\lambda)$.

Since in general the spectrum $\sigma(G)$ of $G$ consists of a point part, $\sigma_p(G)$, and a continuous part, $\sigma_c(G)$ (with $\sigma(G)=\overline{\sigma_p(G)}\cup\sigma_c(G)$), the QP states can only be expected to have their usual properties (cf Zeldovich 1967) if the continuous part $\sigma_c(G)$ is void. In particular, the QP states can only be complete if the spectrum of $G$ is a pure point spectrum, i.e. if $\sigma(G)=\overline{\sigma_p(G)}$. This is illustrated in the following proposition.

*Proposition 3.* Assume in addition to (A1) and (A2(a)–(f)) that $G$ has a pure point spectrum $\sigma(G)=\overline{\sigma_p(G)}=\overline{\{\lambda_n|n=1,2,3,\ldots\}}$ and that the corresponding orthonormal eigenvectors $\psi(\lambda_n)$ of $G$ fulfil $\psi(\lambda_n)\in\mathrm{dom}[H(0)]$, $n=1,2,3,\ldots$. Then the QP states corresponding to $H(t)$ are given by $\Psi(\lambda_n,t)=U(t)\psi(\lambda_n)$ and the following relations hold.

(a) For each fixed $t\in R$ $\{\Psi(\lambda_n,t)|n=1,2,3,\ldots\}$ constitute a complete orthonormal system in $\mathscr{H}$.

(b) If $\Phi(t)\in\mathrm{dom}[H(t)]$ is a solution of (2) then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(\lambda_n,t),\Phi(t))=0 \qquad \text{for all } n=1,2,3,\ldots.$$

(c) For all $\Phi\in\mathscr{H}$

$$|(\Psi(\lambda_n,t+\tau),\Phi)|=|(\Psi(\lambda_n,t),\Phi)|$$

and

$$(\Psi(\lambda_n,t),\Phi)=\mathrm{e}^{\mathrm{i}t\lambda_n}p_n(t), \qquad p_n(t+\tau)-p_n(t), \qquad n=1,2,3,\ldots.$$

(d) If $\lambda_k\tau=\lambda_j\tau+2\pi n$, $n\in Z$, then any superposition of $\Psi(\lambda_j,t)$ and $\Psi(\lambda_k,t)$ belongs to the same quasi-energy.

*Proof.* (a) follows from the fact that $\{\Psi(\lambda_n)|n=1,2,3,\ldots\}$, as eigenvectors of a self-adjoint Hamiltonian with pure point spectrum, span all of $\mathscr{H}$, and from the unitarity of $U(t)$. (b) is proved by noting

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(\lambda_n,t),\Phi(t))=\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(\lambda_n,0),\Phi(0))=0.$$

(c) is implied by

$$|(\Psi(\lambda_n,t+\tau),\Phi)|=|\mathrm{e}^{\mathrm{i}\lambda\tau}(\Psi(\lambda_n,t),\Phi)|$$

and by

$$(\Psi(\lambda_n,t),\Phi)=(P(t)\,\mathrm{e}^{-\mathrm{i}tG}\psi(\lambda_n),\Phi)=\mathrm{e}^{\mathrm{i}t\lambda_n}(P(t)\psi(\lambda_n),\Phi).$$

(d) is clear since quasi-energies are defined up to multiples of $2\pi n/\tau$, $n\in Z$.

Propositions 2 and 3 contain a subtle point which has not been realised before. In fact one cannot rule out the (pathological) case that $G$ has eigenstates $\phi$ which are not in the domain of $H(0)$. In such a case $U(t)\phi$ is no QP state of $H(t)$ and the system of QP states is not complete. This shows that, in contrast to the conjecture to be found in Salzman (1974a), the correspondence between eigenstates of $G$ (eigenvalues of $G$) and QP states of $H(t)$ (quasi-energies of $H(t)$) is in general not one to one.

At this point it is easy to understand why QP states always exist and fulfil (a)–(d) of proposition 3 if $H(t)$ is strongly continuous in $t$ and $\mathcal{H}$ is finite dimensional. In this case (A1) and (A2(a)–(f)) are fulfilled and $G$ is compact and unitarily equivalent to a Hermitian matrix in $C^m$ for some $m \in N$ (Reed and Simon 1975). Consequently $G$ has a pure point spectrum (possibly degenerate), $\sigma(G) = \sigma_p(G) = \{\lambda_1, \ldots, \lambda_m\}$.

If the spectral condition in propositions 3 is not satisfied, i.e. the continuous spectrum of $G$ is non-empty, $\sigma_c(G) \neq \emptyset$, then clearly proposition 3(a) does not hold, whereas propositions 3(b)–(d) remain valid for each $\lambda_n \in \sigma_p(G)$ without any change.

As an example we consider a particle of charge $-e$ and mass $m$ in a spherically symmetric potential under the influence of an external wave field ($\hbar = 1$).

We choose

$$\mathcal{H} = L^2(R^3), \qquad H = -\frac{\Delta}{2m} + \frac{m\nu^2}{2}|\boldsymbol{x}|^2, \qquad \mathrm{dom}(H) = \mathrm{dom}(\Delta) \cap \mathrm{dom}(|\boldsymbol{x}|^2),$$

$$\boldsymbol{A}(t, \boldsymbol{x}) = 2a\cos(\omega x_3/c)(\cos\omega t, -\sin\omega t, 0), \qquad \omega < \nu,$$

$$V(t, \boldsymbol{x}) \equiv \frac{e}{mc}\boldsymbol{A}(t, \boldsymbol{x}) \cdot \frac{1}{\mathrm{i}}\boldsymbol{\nabla} + \frac{e^2}{2mc^2}\boldsymbol{A}^2(t, \boldsymbol{x}),$$

$$H(t) = \frac{1}{2m}\left(\frac{1}{\mathrm{i}}\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{A}(t)\right)^2 + \frac{m\nu^2}{2}|\boldsymbol{x}|^2 = H + V(t), \qquad \mathrm{dom}[H(t)] = \mathrm{dom}(H).$$

Then $U(t)$ takes on the Floquet form (Salzman 1947b, Mitter and Pötz 1981)

$$U(t) = \mathrm{e}^{\mathrm{i}\omega t L_3}\,\mathrm{e}^{-\mathrm{i}tG},$$

$$G = H + \omega L_3 + \frac{2e^2a^2}{mc^2}\cos^2\left(\frac{\omega}{c}x_3\right) + 2\frac{ea}{mc}\cos\left(\frac{\omega}{c}x_3\right)\frac{1}{\mathrm{i}}\frac{\partial}{\partial x_1}$$

$$= H + \omega L_3 + V(0), \qquad \mathrm{dom}(G) = \mathrm{dom}(H),$$

where $L_3$ abbreviates the third component of the angular momentum operator. That $G$ is self-adjoint and $\mathrm{dom}(G) = \mathrm{dom}(H)$ follows from (Reed and Simon 1975)

$$\|\omega L_3(H + \alpha 1)^{-1}\| = \sup_{\substack{n,l=0,1,2,\ldots \\ |l_3| \leq l}} \left|\frac{\omega l_3}{\nu(2n + l + 3/2) + \alpha}\right| = \frac{\omega}{\nu} < 1, \qquad \alpha \geq 0,$$

and from the fact that

$$V(0)(H + \omega L_3 + z)^{-1}, \qquad z \in C - R,$$

is easily seen to be a compact operator in $L^2(R^3)$. The latter result also implies the emptiness of the essential spectrum $\sigma_{\mathrm{ess}}(G)$ since

$$\sigma_{\mathrm{ess}}(G) = \sigma_{\mathrm{ess}}(H + \omega L_3) = \emptyset.$$

Thus the dominance of $m\nu^2|\boldsymbol{x}|^2/2$ for $|\boldsymbol{x}| \to \infty$ forces the spectrum of $G$ to be purely discrete, i.e. $\sigma(G)$ consists only of isolated eigenvalues of finite multiplicity, and all assumptions of proposition 3 are satisfied in this case. These results are essentially independent of the special form of the potential $m\nu^2|\boldsymbol{x}|^2/2$ and also hold if $m\nu^2|\boldsymbol{x}|^2/2$ is replaced by some $V(\boldsymbol{x})$ fulfilling $V(\boldsymbol{x}) \in L^1_{\mathrm{loc}}(R^3)$, $V(\boldsymbol{x}) \geq m\nu^2|\boldsymbol{x}|^2/2 + c$, $c \in R$. If the potential does not dominate (diverge) for $|x| \to \infty$, then $G$ will in general exhibit a continuous part in its spectrum and thus the QP states will lose their completeness property. In this case $\omega L_3$ is by no means a small perturbation of $H$, and also stability assertions regarding the spectra of $H + \omega L_3$ and $G$ cannot be obtained by the simple methods described above.

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